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Some aspects of non-uniform convergence in an elliptic singular perturbation problem

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SUMMARY
In elliptic singular perturbation problems several types of boundary layers occur. A model problem is investigated and it is shown that, by extension of the familiar stretching technique to parameters, uniform results can be obtained.

1. Introduction

In many singular perturbation problems (with a small parameter ε), it occurs that the actual construction of an asymptotic approximation of the solution depends discontinuously on a certain additional parameter μ. In most of such cases the exact solution is continuous or even analytical in μ. Turning point problems like

$$\varepsilon \frac{d^2 u}{dx^2} + \mu x \frac{du}{dx} + g(x)u = 0 , \qquad u(\pm 1) \text{ prescribed} ,$$

are a well-known example of this behaviour.

The phenomenon is particularly apparent in the Dirichlet problem for a second order linear differential equation of elliptic type,

$$(\varepsilon L_2 + L_1)\Phi = 0 , \qquad 0 < \varepsilon \ll 1 , \tag{1.1}$$

in a convex domain $D \subset \mathbb{R}^2$. The solution of (1.1) exhibits a boundary layer behaviour and it is well-known that the position and the type of the boundary layers depend on the characteristics of the first order differential operator L_1. Here the parameter μ is a measure for the angle between these characteristics and the boundary of D.

It is our opinion that the phenomenon is due to non-uniform convergence with respect to μ. Consequently we want to attack the problem by applying the stretching technique to the parameter μ. As the solution of a formal limit equation we obtain a function that constitutes the missing link for an asymptotic approximation uniform in μ. This is shown by means of suitable limit processes.

Already two papers ([1], [4]) have been devoted to this subject, so a special justification seems in place. Firstly we treat the transition from parabolic- to free boundary layer which has not been done so far. In the second place the asymptotic analysis of Comstock is improved somewhat and an important matching result which was not mentioned by Grasman is given.

Throughout this paper we will limit ourselves to first order asymptotic approximations. It is a rather straightforward but tedious procedure to extend the analysis to higher order approximations.

2. The boundary value problem

Instead of considering the general equation (1.1), we investigate an equation with constant coefficients which is frequently treated as a model problem. As a further simplification we take as domain the quarter plane

$$D = \{(x, y) \mid x > 0, \, y > 0\} . \tag{2.1}$$

In this manner, uninteresting technical difficulties are eliminated and it becomes possible to emphasize the transition of the various boundary layers. Thus the problem takes the form

$$L_{\varepsilon,\mu}[\Phi] = (\varepsilon\Delta - D_\mu)[\Phi] = 0\,, \qquad 0<\varepsilon \ll 1\,, \tag{2.2}$$

$$\begin{cases} \Phi(x,0) = 0\,, & x>0\,,\\ \Phi(0,y) = f(y)\,, & y>0\,,\\ f \text{ continuous and bounded for } y\to\infty,\ f(0)\neq 0\,. \end{cases} \tag{2.3}$$

Here Δ stands for the Laplace operator in two dimensions and D_μ for a first order differential operator

$$D_\mu = \mu\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} + c\,, \qquad c \geqq 0\,. \tag{2.4}$$

In order to ensure uniqueness we have to impose a complicated condition upon Φ concerning its growth at infinity and its behaviour near the corner point, where the boundary conditions are discontinuous. In [2] both this condition and a representation of the exact solution are given.

Of course we can reduce the number of constants in the operator D_μ by scaling the parameters ε and μ. However, it is our intention to obtain a clear view on the manner in which the coefficients of the equation occur in the expressions for an approximate solution.

The method of matched asymptotic expansions can be succesfully applied to this problem (see for instance [3], [6]). It turns out that a crucial role is played by the slope of the characteristics of the reduced equation $D_\mu[\Phi]=0$. These are the lines

$$x = \frac{\mu}{b}\,y + \text{constant}\,. \tag{2.5}$$

We take b as a fixed positive number and μ as a parameter. For a first order asymptotic approximation three cases are to be distinguished (see [3], where (i) and (ii) are treated):

(i) $\mu<0$: ordinary boundary layer along $x=0$, $\Phi\sim Q$ with

$$Q(x,y) = f(y)\,e^{\mu(x/\varepsilon)}\,. \tag{2.6}$$

(ii) $\mu=0$: parabolic boundary layer along $x=0$, $\Phi\sim P$ with

$$P(x,y) = 2(\pi)^{-\frac{1}{2}}\int_{(bx^2/4\varepsilon y)^{1/2}}^{\infty} \exp\left(-\tau^2 - \frac{cx^2}{4\varepsilon\tau^2}\right) f\left(y - \frac{bx^2}{4\varepsilon\tau^2}\right) d\tau\,. \tag{2.7}$$

(iii) $\mu>0$: free boundary layer along the characteristic line $bx=\mu y$. Since, for the boundary value problem we consider, this case is not treated in the literature we give some details. The solution of the reduced equation $D_\mu[\Phi]=0$ can satisfy all boundary conditions,

$$\begin{cases} F_1(x,y) = 0\,, & \mu y - bx < 0\,,\\ F_2(x,y) = f\left(\dfrac{\mu y - bx}{\mu}\right) e^{-(c/\mu)x}\,, & \mu y - bx > 0\,. \end{cases} \tag{2.8}$$

The discontinuity ($f(0)\neq 0$) propagates along the characteristic line through the origin. Yet from the ellipticity of (2.2) we expect the solution to be smooth throughout D. So there has to be a boundary layer along $bx=\mu y$. Stretching of a coordinate orthogonal to this line and a careful analysis of the differential equation and of the matching conditions with F_1 and F_2 leads to the local approximation

$$F_3(x,y) = \tfrac{1}{2}f(0)\,e^{-(c/b)y}\,\mathrm{erfc}\left(\frac{bx-\mu y}{2((\mu^2+b^2)b^{-1}\varepsilon y)^{\frac{1}{2}}}\right). \tag{2.9}$$

Now a uniform approximation can easily be constructed: $\Phi\sim F$ with

$$F(x,y) = \tfrac{1}{2}f\left(\frac{\mu y - bx}{\mu}\right) e^{-(c/\mu)x}\,\mathrm{erfc}\left(\frac{bx-\mu y}{2((\mu^2+b^2)b^{-1}\varepsilon y)^{\frac{1}{2}}}\right). \tag{2.10}$$

Using a maximum principle one can show that the exact solution is continuous in μ, uniform in every bounded subdomain of D (the problem is regular with respect to μ; see [2]). But it is obvious that by putting $\mu=0$ in $Q(x, y)$ or $F(x, y)$, one does not arrive at $P(x, y)$. It is this paradox that we will analyse in the next sections.

3. Non-uniform convergence with respect to a parameter

An important tool in singular perturbation techniques is that of a special limit process defined in the following way. Let $\delta(\varepsilon)$ be an order function (a real continuous function, defined and positive on $(0, \varepsilon_0)$ and such that $\lim_{\varepsilon\downarrow 0}\delta(\varepsilon)$ exists), and x a variable. By stretching of x we mean the introduction of a local variable ξ_δ by means of $x=\xi_\delta\delta(\varepsilon)$. Now the ξ_δ-limit of a function $\phi(x, \varepsilon)$ is defined by

$$\lim_{\xi_\delta}\phi(x, \varepsilon) = \lim \phi \text{ for } \varepsilon\downarrow 0 \text{ with } \xi_\delta \text{ fixed}.$$

Similar definitions can be given for functions of several variables (see [5] for more details).

The solution of the problem (2.2)–(2.3) depends, besides on ε, on x, y and μ. The distinction between variables and parameters stems from the manner in which they occur in the differential equation that implicitly defines the function, but this distinction is not inherent in the nature of the function. This motivates our attempt to solve the observed paradox by applying the method of local variables and limit processes to the parameter μ as well.

The following example of an explicitly given function indicates the kind of results we are looking for. Consider

$$\psi(x, \mu, \varepsilon) = \exp\left(-\frac{\mu x}{\varepsilon} - \frac{x}{\varepsilon^{\frac{1}{2}}}\right) + \exp\left(-\frac{1}{\varepsilon^2}\right), \quad 0<\varepsilon\ll 1,\ 0\leqq\mu<\infty,\ 0\leqq x<\infty. \tag{3.1}$$

Putting $x=\xi_\delta\delta(\varepsilon)$ and taking the ξ_δ-limit of ψ we find

$$\mu\neq 0:\ \lim_{\xi_\delta}\psi(x, \mu, \varepsilon) = e^{-\mu\xi_\delta} \text{ with } \delta(\varepsilon)=\varepsilon,$$
$$\mu=0:\ \lim_{\xi_\delta}\psi(x, 0, \varepsilon) = e^{-\xi_\delta} \text{ with } \delta(\varepsilon)=\varepsilon^{\frac{1}{2}}.$$

One easily proves that in both cases the limit function is an asymptotic approximation which is uniformly valid in x. But certainly neither of them is valid uniformly in μ, and in fact they do not even match with respect to μ. The limit $\mu\downarrow 0$ and the ξ_δ-limits are not interchangeable. The remedy is to introduce a local parameter v_γ by means of $\mu=v_\gamma\gamma(\varepsilon)$, and to take the ξ_δ, v_γ-limit of ψ

$$\lim_{\xi_\delta, v_\gamma}\psi(x, \mu, \varepsilon) = \exp(-v_\gamma\xi_\delta-\xi_\delta) \text{ with } \delta(\varepsilon)=\varepsilon^{\frac{1}{2}} \text{ and } \gamma(\varepsilon)=\varepsilon^{\frac{1}{2}}.$$

This limit function is an asymptotic approximation which is uniformly valid in x and in μ.

For the explicitly given function (3.1) this procedure is obvious. In the next section we will show that the same procedure works for the function implicitly defined by (2.2)–(2.3).

4. The generalized boundary layer function

From the differential equation (2.2) it is clear that the only significant stretching of the variable x and the parameter μ is such that both $\varepsilon\partial^2/\partial x^2$ and $\mu\partial/\partial x$ become of formal order one. In this way we obtain the limit equation

$$\varepsilon\frac{\partial^2 G}{\partial x^2} - \mu\frac{\partial G}{\partial x} - b\frac{\partial G}{\partial y} - cG = 0. \tag{4.1}$$

The solution that meets the conditions

$$G(x, 0) = 0,\quad G(0, y) = f(y), \tag{4.2}$$

is given by the *generalized boundary layer function*

$$G(x, y) = 2(\pi)^{-\frac{1}{2}} e^{\mu x/2\varepsilon}\int_{(bx^2/4\varepsilon y)^{\frac{1}{2}}}^{\infty} \exp\left(-\tau^2 - \frac{cx^2}{4\varepsilon\tau^2} - \frac{\mu^2x^2}{16\varepsilon^2\tau^2}\right) f\left(y - \frac{bx^2}{4\varepsilon\tau^2}\right) d\tau. \tag{4.3}$$

Our main result is the following. The generalized boundary layer function contains (with respect to μ) the ordinary boundary layer function and the parabolic boundary layer function and it matches the free boundary layer function. Moreover, we will show that $G(x, y)$ contains hidden boundary layers which were pointed out by N. M. Temme ([7]) in his direct asymptotic analysis of the integral representation of the exact solution of a similar problem.

To prove these results, we have to give an asymptotic expansion of G for $\varepsilon\downarrow 0$ under various assumptions for x, y and μ. As a first remark, we mention that by simply putting $\mu=0$ in (4.3), we obtain the parabolic boundary layer function (2.7), so this part of the statement needs no further comment.

5. Asymptotic analysis of the generalized boundary layer function

With the definitions

$$\alpha = \frac{\mu^2 x^2}{16} + \frac{\varepsilon c x^2}{4}, \quad \beta = \tfrac{1}{2}x\left(\frac{b}{y}\right)^{\frac{1}{2}}, \tag{5.1}$$

and with $\tau\to\tau\varepsilon^{-\frac{1}{2}}$, formula (4.3) can be written as

$$G(x, y) = 2(\pi\varepsilon)^{-\frac{1}{2}}\, e^{\mu x/2\varepsilon} \int_{\beta}^{\infty} \exp\left\{\left(-\tau^2 - \frac{\alpha}{\tau^2}\right)\varepsilon^{-1}\right\} f\left(y - \frac{bx^2}{4\tau^2}\right) d\tau\,. \tag{5.2}$$

For $\varepsilon\downarrow 0$ the asymptotic behaviour of the integral is determined by the stationary points of the integrand. We note that there are two real stationary points $\pm\tau_0$, where $\tau_0=\alpha^{\frac{1}{4}}$. Since $\beta>0$, it would seem that only the contribution of the positive stationary point needs to be considered. However, for $\alpha\downarrow 0$ the effect of $-\tau_0$ will increase, and a significant contribution in a special part of the domain should be expected.

After the transformation

$$\tau - \frac{\alpha^{\frac{1}{2}}}{\tau} = q\,, \tag{5.3}$$

we write (5.2) as a product

$$G(x, y) = I(x) J(x, y)\,, \tag{5.4}$$

where the factors are given by

$$I(x) = \exp\left(\frac{\mu x}{2\varepsilon} - \frac{2\alpha^{\frac{1}{2}}}{\varepsilon}\right) \tag{5.5}$$

and

$$J(x, y) = 2(\pi\varepsilon)^{-\frac{1}{2}} \int_{\beta-(\alpha^{1/2}/\beta)}^{\infty} \exp\left(-\frac{q^2}{\varepsilon}\right) f\big(y - R(q)\big)\, \frac{d\tau}{dq}\, dq\,, \tag{5.6}$$

with

$$R(q) = \frac{2\beta^2 y}{q^2 + q(q^2+4\alpha^{\frac{1}{2}})^{\frac{1}{2}} + 2\alpha^{\frac{1}{2}}}\,. \tag{5.7}$$

From (5.3) we obtain the explicit form

$$\frac{d\tau}{dq} = \frac{1}{2} + \frac{q}{2(q^2+4\alpha^{\frac{1}{2}})^{\frac{1}{2}}}\,. \tag{5.8}$$

Up to now we have assumed that the values of x and y are restricted to those corresponding with the domain D. However, we may include negative values of x in the asymptotic analysis of the functions $I(x)$ and $J(x, y)$ and in this way more insight can be gained as we will see below.

The factorization (5.4) is chosen in order to emphasize that two different asymptotic approximations are to be made. So at first we treat $I(x)$ and $J(x, y)$ independently.

For $\varepsilon\downarrow 0$ it follows from (5.5) and the definition of α (5.1) that

$$I(x) \sim \begin{cases} e^{-cx/\mu} & \text{for} \quad \mu x > 0 \,, \\ e^{\mu x/\varepsilon} & \text{for} \quad \mu x < 0 \,. \end{cases} \tag{5.9}$$

From (5.6), it is clear that for $\varepsilon \downarrow 0$ the asymptotic behaviour of $J(x, y)$ depends on whether $q=0$ lies within the interval of integration or not. In other words, on the sign of

$$\beta - \frac{\alpha^{\frac{1}{2}}}{\beta} = \frac{bx - |\mu| \dfrac{|x|}{x} y}{2(by)^{\frac{1}{2}}} + O(\varepsilon) \,.$$

Straightforward application of the method of Laplace leads to the result

$$J(x, y) \sim \begin{cases} f\left(\dfrac{|\mu|\, y - b|x|}{|\mu|}\right) & \text{for} \quad b|x| < |\mu|\, y \,, \\ 0 & \text{for} \quad b|x| > |\mu|\, y \,. \end{cases} \tag{5.10}$$

In order to examine the transition, we take the t_δ-limit of $J(x, y)$, where t_δ is defined by $b|x| - |\mu|\, y = t_\delta \delta(\varepsilon)$. We find

$$\lim_{t_\delta} J(x, y) = \tfrac{1}{2} f(0) \operatorname{erfc}\left(\frac{t_\delta}{2(by)^{\frac{1}{2}}}\right), \quad \text{with} \quad \delta(\varepsilon) = \varepsilon^{\frac{1}{2}} \,. \tag{5.11}$$

Obviously the function $J(x, y)$ exhibits a boundary layer behaviour. The boundary layer corresponding to the stationary point $+\tau_0$ is situated around the line $bx = |\mu|\, y$. There is another boundary layer corresponding to the stationary point $-\tau_0$ and situated around the line $bx = -|\mu| y$. Referring to the domain D we call the first one internal and the second one external.

We observe that the asymptotic order of $I(x)$ depends rather strongly on the sign of μ. In contrast to this we note from (5.6) that $J(x, y)$ does not depend on the sign of μ at all and this is reflected in the results (5.10) and (5.11). Combining the approximations for $I(x)$ and $J(x, y)$ obtained above, we find for the inside of D (i.e. $x > 0$):

(i) $\mu < 0$: ordinary boundary layer along $x = 0$

$$G(x, y) \sim e^{\mu x/\varepsilon} f\left(\frac{|\mu|\, y - bx}{|\mu|}\right) \sim f(y)\, e^{\mu(x/\varepsilon)} \,. \tag{5.12}$$

The free boundary layer along $bx = |\mu|\, y$ is hidden by the multiplicative function $I(x)$.

(ii) $\mu > 0$: free boundary layer along $bx = |\mu|\, y$

$$G(x, y) \sim \begin{cases} f\left(\dfrac{\mu y - bx}{\mu}\right) e^{-(c/\mu)\, x} \,, & \mu y - bx > 0 \,, \\ \tfrac{1}{2} f(0)\, e^{-(c/b) y} \operatorname{erfc}\left(\dfrac{bx - \mu y}{2(by\varepsilon)^{\frac{1}{2}}}\right); & \mu y - bx = O(\varepsilon^{\frac{1}{2}}) \,, \\ 0 \,, & \mu y - bx < 0 \,. \end{cases} \tag{5.13}$$

Comparing these results with the formulae (2.6), (2.8) and (2.9), we conclude that our main assertion, stated in Section 4, is indeed correct.

Similar results can be obtained for $x < 0$, but instead of stating them in detail we prefer to present an overall picture in figure 1 and 2.

Another important observation has to be made. The influence of the external free boundary layer is negligible in the domain D, except for an (ε-dependent) neighbourhood of the origin. To calculate the contribution, we have to take the ξ_δ, η_γ-limit of $G(x, y)$ ((5.4)), where $x = \xi_\delta \delta(\varepsilon)$ and $y = \eta_\gamma \gamma(\varepsilon)$ are so defined that $\beta - \alpha^{\frac{1}{2}} \beta^{-1}$ is of formal order one. Then $f(y - R(q))$ can be replaced by $f(0)$ (the first term in the Taylor expansion). But now the second term of $d\tau/dq$ cannot be neglected (which is another way of saying that the stationary point $-\tau_0$ has to be taken into account). In fact we find

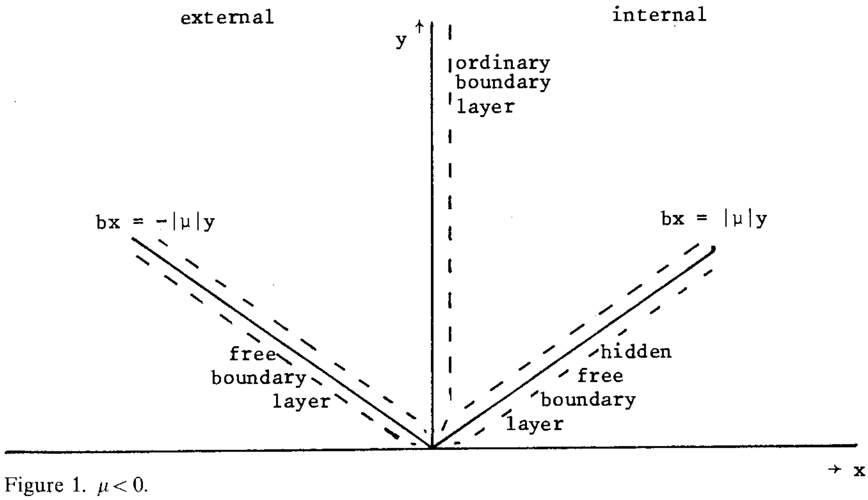

Figure 1. $\mu < 0$.

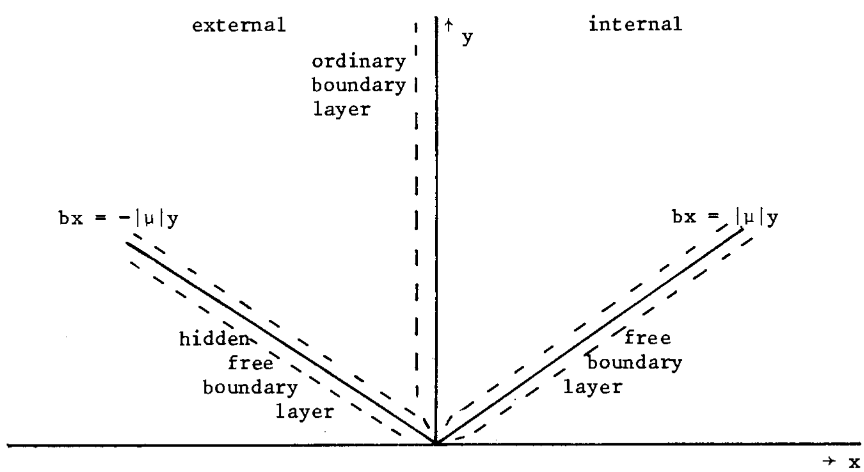

Figure 2. $\mu > 0$.

$$\lim_{\xi_\delta, \eta_\gamma} G(x, y) = \lim_{\xi_\delta, \eta_\gamma} \tfrac{1}{2} f(0) \left\{ \exp\left(\frac{\mu x - 4\alpha^{\frac{1}{2}}}{2\varepsilon}\right) \operatorname{erfc}\left(\beta\varepsilon^{-\frac{1}{2}} - \alpha^{\frac{1}{2}}\beta^{-1}\varepsilon^{-\frac{1}{2}}\right) \right.$$

$$\left. + \exp\left(\frac{\mu x + 4\alpha^{\frac{1}{2}}}{2\varepsilon}\right) \operatorname{erfc}\left(\beta\varepsilon^{-\frac{1}{2}} + \alpha^{\frac{1}{2}}\beta^{-1}\varepsilon^{-\frac{1}{2}}\right) \right\}. \tag{5.14}$$

We have to distinguish the two cases $\mu < 0$ and $\mu > 0$, and, although each term of (5.14) gives a different contribution in each case, the total outcome is the same, due to a certain symmetry:

$$\lim_{\xi_\delta, \eta_\gamma} G(x, y) = \tfrac{1}{2} f(0) \left\{ \operatorname{erfc}\left(\frac{bx - \mu y}{2(by\varepsilon)^{\frac{1}{2}}}\right) + e^{\mu x/\varepsilon} \operatorname{erfc}\left(\frac{bx + \mu y}{2(by\varepsilon)^{\frac{1}{2}}}\right) \right\}. \tag{5.15}$$

One easily verifies that this limit function matches, with respect to μ, the corresponding limits of $Q(x, y)$, $P(x, y)$ and $F(x, y)$ ((2.6), (2.7) and (2.10)). In [2] and [4] the same limit function was found with the aid of stretching of x, y and μ, and a detailed examination of the differential operator and the boundary and matching conditions.

6. Concluding remarks

The formal procedure of coordinate stretching and matching does not imply that the obtained functions are indeed asymptotic approximations of the exact solution. This can be verified by a proof in which the maximum principle is the main tool. For this aspect of the problem we refer to [3] and [6].

However, once more it appears that the formal way of action yields the right results. By purely formal manipulation we did obtain the limit equation (4.1). The structure of its solution (4.3) is much simpler than the structure of the exact solution of (2.2)–(2.3). Nevertheless we were able to show that all significant features of the exact solution are contained in (4.3) for all values of μ.

In the asymptotic analysis of the generalized boundary layer function (4.3) the decomposition (5.4) is an essential step. The factors $I(x)$ and $J(x, y)$, defined in (5.5) and (5.6), both admit a simple treatment with a perspicuous outcome. Here is again an illustration of the fact that multiplication of two asymptotic approximations can be a source of confusing results.

Our main purpose was to show that the observed discontinuity in the asymptotic approximation can be interpreted as non-uniform convergence with respect to a certain parameter and that the method of stretching works in this case as well. It is in this sense that we expect that our treatment of a rather special problem may have wider application.

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